

All order covariant tubular expansion

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Abstract

We consider tubular neighborhood of an arbitrary submanifold embedded in a (pseudo-)Riemannian manifold. This can be described by Fermi normal coordinates (FNC) satisfying certain conditions as described by Florides and Synge in [7]. By generalizing the work of Muller *et al* in [22] on Riemann normal coordinate expansion, we derive all order FNC expansion of vielbein in this neighborhood with closed form expressions for the curvature expansion coefficients. Our result is shown to be consistent with certain integral theorem for the metric proved in [7]. We explain how this result is relevant in understanding an interacting multi-particle system forming bound states in a curved background and a loop space description of non-linear sigma model.

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1 Motivation and summary

As it is well known, Riemann normal coordinate (RNC) system is a very useful tool in differential geometry and general relativity [1, 2]. In particular, it is used for computing covariant Taylor expansion of tensors around a point in a manifold [3, 4].

Fermi normal coordinate (FNC) system is a generalization of RNC in the sense that the RNC-origin, which may be viewed as a zero-dimensional submanifold of the ambient space, is replaced by a higher dimensional one¹. It is related to a general coordinate system through the exponential map along the normal directions. The region surrounding the submanifold where FNC is well defined, i.e. where the exponential map is a diffeomorphism, is called a *tubular neighborhood*. The existence theorem for such a neighborhood suffices as a powerful tool for various analysis in differential geometry [8].

The use of FNC in general relativity has been mainly due to the fact that one wishes to study various physical effects around a particle worldline (one dimensional submanifold) in a given background. A partial list of references is [9, 10, 11, 14, 15]. In all such applications one needs to know explicitly the covariant Taylor expansion coefficients of geometric

¹FNC was first introduced by E. Fermi for a curve in [5]. It was then followed by various other generalizations (see [6] and references therein). Our consideration, which is same as that of [7], will be described in detail in §3.

quantities around the worldline/submanifold². So far only a few such coefficients have been known.

In one line of research [14] FNC expansion around a worldline has been used to compute curvature corrections to the energy spectrum of hydrogen-like atoms in curved backgrounds. The reason for considering a worldline is the following approximation (see also [15]): the center of mass (CM) of the system falls freely along a geodesic during the time interval relevant for the study of the system (e.g. an atomic transition). This is of course a fair assumption when the CM mass is large. However, a framework for a more complete treatment needs to be investigated. For example, one may wonder how the internal dynamics of the atom (or any other composite object) backreacts to the motion of the CM. In other words, if it should be possible to derive the geodesic equation for CM at the leading order in a semi-classical expansion³ where the CM mass is large.

To answer the above question one needs a fully relativistic treatment of a multi-particle interacting system forming a bound state. This is a difficult problem [19]. However, it is interesting to note that in spirit it is very similar to the computation of α' corrections in string theory, though there are differences in details (of course). Here the analogue of the composite system is a closed string to be viewed as a bound configuration of an infinite number of particles. The relativistic theory governing the system is given by the two-dimensional non-linear sigma model (NLSM). The analogy can be seen very explicitly in a loop space description studied recently in [20, 21]. In particular, in [21] a semi-classical limit ($\alpha' \rightarrow 0$) with the following property has been discussed. If the target space is M then the semi-classical expansion is related to the tubular expansion of NLSM around the embedding $M \hookrightarrow LM$, where LM is the (infinite dimensional) loop space and the submanifold, which is isomorphic to M , is the space of all vanishing loops. The latter turns out to be the space of all possible positions of CM of the loop. The internal dynamics is described in the tubular neighborhood and the effective theory of the CM in M is obtained by using a Born-Oppenheimer type approximation.

Since α' corrections to the effective theory needs to be derived from the tubular expansion of NLSM, it is important to know the Riemannian structure in the relevant tubular neighborhood. It was shown in [21] that such a Riemannian structure is related to \mathbf{R}_{TM} ,

²The same is true for certain applications outside physics also. See [16] which describes how this is used in computing volume of tubes [17, 18].

³What we have in mind is an expansion in inverse of the CM-mass. The quantum mechanical \hbar may be set to 1.

i.e. a particular Riemannian structure on TM that views the zero section $TM_0(\cong M)$ as an embedded submanifold admitting a tubular neighborhood.

In §2 we will discuss how understanding \mathbf{R}_{TM} is also a problem relevant to the original multi-particle system in M . Moreover, we will describe two steps following which \mathbf{R}_{TM} may be derived completely. The first step is to have an understanding of tubular expansion in a generic case where one considers an arbitrary submanifold embedded in a (pseudo-)Riemannian ambient space. The second step is a mathematical problem [23] that we will make precise in §2 itself. In this work we would like to solve the first step. In particular, we will derive all order covariant Taylor expansion of the vielbein components in such a generic case. Using this result one can compute the metric-expansion to any desired order. In fact, we will present explicit results up to 10-th order. This enables one, in particular, to extend the analysis done for the general relativistic applications by considering particle worldline as mentioned earlier to higher orders.

We now briefly discuss the technical points relevant to our analysis towards pursuing the first step mentioned above. In [7], Florides and Synge (FS) constructed the special coordinate system under consideration for an arbitrary submanifold embedded in a higher dimensional (pseudo-)Riemannian space⁴. They wrote down the special coordinate conditions in terms of the metric and proved an integral theorem describing its behavior away from the submanifold. We will review the basic results in §3.

In [22], Muller, Schubert and van de Ven considered the RNC coordinate conditions written in terms of the vielbein and spin connection. Using certain differential geometric techniques it was possible to write down an integral equation for the vielbein in terms of the Riemann curvature tensor⁵. The authors were able to solve this equation to produce the complete RNC expansion of vielbein with a closed form covariant expression for the curvature expansion coefficients.

Here we will use the same techniques to generalize the results of [22] to the case of FNC. In particular, we derive the integral equation for the vielbein and closed form covariant expressions for its curvature expansion coefficients in the set up considered in [7]. The results are different when the vector index of the vielbein takes values along the directions

⁴The authors of [7] called it a *submanifold based coordinate system*. However, following the modern nomenclature (see, for example [16]) we will continue to call it FNC.

⁵As mentioned in [22], such coordinate conditions are the gravity analogue of the Fock-Schwinger gauge [24] in gauge theory which can also be used to write down an integral equation for the gauge potential in terms of the field strength [25].

tangential and transverse to the submanifold. The transverse results are exactly the same as that of [22], as expected. This will be discussed in §4. In §5, we show how our result is consistent with the metric integral theorem of [7]. Explicit numerical results for the expansion of vielbein and metric up to 10-th order in FNC have been presented in one of the appendices.

2 On the relevance of \mathbf{R}_{TM}

Here we will explain how \mathbf{R}_{TM} is relevant to the study of a multi-particle interacting system forming a bound state in a curved space. This will also specify \mathbf{R}_{TM} more precisely. Then we will explain how the problem of determining \mathbf{R}_{TM} is related to the result of tubular expansion (see §4.2) in a generic case found in this work.

We begin by considering a system of two interacting particles in a curved space, say M . Extension of the argument to higher number of particles is straightforward. We assume that there exists a generally covariant quantum mechanical theory which efficiently describes the bound state and that the configuration space⁶ of this theory is a normal neighborhood of $M \times M$ ⁷ which corresponds to having both the particles in a single normal neighborhood of M . The latter implies, in particular, that the diagonal Δ of $M \times M$ intersects with the configuration space.

We now argue that the CM lies in the diagonal submanifold $\Delta(\cong M) \hookrightarrow M \times M$. This is evidently true when the two particles coincide in M . When the particles are separated, one may proceed as follows. Given a two-particle configuration, erect a tangent space $T_x M$ such that x lies in the neighborhood of that configuration. Then find the preimage of the configuration in $T_x M$ under the exponential map $\exp_x : T_x M \rightarrow M$. If the sum of the position vectors (in the preimage) in $T_x M$ vanishes, then x is the CM. Therefore all possible configurations can be described on the tangent bundle TM in the following way. The CM always lies on the zero section $TM_0(\cong M)$ whereas the relative position is given by an element in the fibre positioned at the CM in TM_0 . Notice that in this description any given two-particle configuration lies entirely on the same fibre, which is not true in

⁶We mean extended configuration space, as M includes time.

⁷In the following discussion we will repeatedly encounter such spaces. For simplicity and to avoid clutter of notations, we will refrain from emphasizing each time that our considerations are local, i.e. restricted to a suitable normal/tubular neighborhood.

the original description in $M \times M$. The above procedure defines a bundle map,

$$\Phi : TM \rightarrow M \times M , \quad (2.1)$$

which relates the two descriptions of the configuration space. It is a diffeomorphism (given that exponential map is a diffeomorphism) and,

$$\Phi : TM_0 \rightarrow \Delta . \quad (2.2)$$

This shows that Δ is the space of all possible CM's.

Given the above discussion, it is now clear that the internal dynamics of the system lives in the tubular neighborhood of $\Delta \hookrightarrow M \times M$. In order to derive the effective theory of the CM in $\Delta(\cong M)$ one may follow the procedure adopted in [21] for NLSM, namely one first expands the generally covariant quantum theory in the relevant FNC and then uses a Born-Oppenheimer type approximation described in that work.

We now specify \mathbf{R}_{TM} precisely. The FNC relevant for the embedding $\Delta \hookrightarrow M \times M$ is related to the general coordinates on $M \times M$ (which gives the usual direct product Riemannian structure) by a coordinate transformation which is precisely given by (2.1). In other words, the Riemannian structure given by the FNC expansion of the vielbein can be thought as the Riemannian structure \mathbf{R}_{TM} that has been induced on TM through the above procedure. It is such that $TM_0(\cong M)$ is viewed as a submanifold of TM which admits a tubular neighborhood/expansion.

We now specify the mathematical problem of finding \mathbf{R}_{TM} completely, given the results of this work. To this end we note the following two points,

- The relevant tubular expansion in TM is a special case of the general form derived in this work. As displayed in §4.2, the expansion coefficients are tensors of the ambient space (in this case TM) evaluated on the submanifold with specific numerical prefactors. Such tensors are various combinations of curvature and its covariant derivatives. These coefficients carry detailed information about the extrinsic properties of the embedding.
- In the generic case considered here, the intrinsic and extrinsic properties of the embedding can be chosen somewhat independently. However, the Riemannian structure of TM is induced from that of M . In other words, there is only one set of

independent geometric data involved in the problem, which is that of M . Therefore, for the embedding $TM_0(\cong M) \hookrightarrow TM$, the tubular expansion coefficients must all be expressed in terms of the intrinsic geometric data of M .

Such relations have been found in [21] up to quadratic order with an undetermined real parameter. Determining \mathbf{R}_{TM} completely is equivalent to finding all such relations. Clearly, the all-order-result found in this work is necessary for such an exercise. This is a work in progress [23].

The rest of the paper is devoted to finding all-order-result for the expansion of vielbein components around an arbitrary submanifold embedding.

3 Metric-integral-theorem due to Florides and Synge

We consider a D -dimensional submanifold M embedded in a higher dimensional pseudo-Riemannian space L of dimension d . Our notation for indices is as follows: Greek indices (α, β, \dots) run over D dimensions, capital Latin indices (A, B, \dots) run over $(d - D)$ transverse dimensions and small Latin indices (a, b, \dots) , over all (d) dimensions. Other notations and conventions essential for the rest of our discussion are presented in appendix A.

The work of [7] proved the existence of a coordinate system $z^a = (x^\alpha, y^A)$ (that will be called FNC following [16]), where x^α is a general coordinate system in M whose embedding is given by,

$$y^A = 0 \ , \tag{3.3}$$

and the following conditions are satisfied for the transverse coordinates,

$$g_{aB}(x, y)y^B = \eta_{aB}y^B \ , \tag{3.4}$$

where $g_{ab}(x, y)$ is the metric tensor in FNC.

The following integral theorem was proved in [7],

$$\begin{aligned} g_{AB}(x, y) &= \eta_{AB} + 2y^C y^D \int_0^1 dt F_1(t) l_{ACDB}(x, ty) \ , \\ g_{A\beta}(x, y) &= y^C \bar{g}_{A\beta, C}(x) + 2y^C y^D \int_0^1 dt F_2(t) l_{ACD\beta}(x, ty) \ , \end{aligned}$$

$$g_{\alpha\beta}(x, y) = G_{\alpha\beta}(x) + y^C \bar{g}_{\alpha\beta, C}(x) + 2y^C y^D \int_0^1 dt F_3(t) l_{\alpha C D \beta}(x, ty) . \quad (3.5)$$

For any function $f(x, y)$ in L , we have defined: $\bar{f}(x) \equiv f(x, 0)$. A comma in the suffix indicates ordinary derivative with respect to the argument. For example,

$$\bar{g}_{\alpha\beta, C}(x) = \lim_{y \rightarrow 0} \frac{\partial}{\partial y^C} g_{\alpha\beta}(x, y) . \quad (3.6)$$

$G_{\alpha\beta} = \bar{g}_{\alpha\beta}$ is the induced metric on M and the functions $F_i(t)$ are defined as follows,

$$F_1(t) = t(1 - t) , \quad F_2(t) = \frac{1}{2}(1 - t^2) , \quad F_3(t) = 1 - t , . \quad (3.7)$$

Finally,

$$l_{acdb}(x, y) = \frac{1}{2}(g_{ab, cd} + g_{cd, ab} - g_{ad, cb} - g_{cb, ad})(x, y) , \quad (3.8)$$

is the linear part of the covariant Riemann curvature tensor.

4 FNC expansion in arbitrary tubular neighbourhood

The above metric-integral-theorem involves the linear part of the curvature tensor. This form is not very useful for deriving the curvature expansion. Below we will use the technique of [22] to derive the curvature expansion for vielbein. In §5 we will show how our result is consistent with that of FS.

4.1 Integral equations for vielbein

We will first derive the integral equations satisfied by the vielbein (eqs.(4.24) below). To this end, we define, following [22], the *radial vector field* (see appendix A for our notations),

$$\mathbf{r} = y^A \mathcal{E}_A . \quad (4.9)$$

Then the coordinate condition (3.4) can be rewritten in terms of the vielbein and the connection one-form in the following way,

$$\mathbf{i}_{\mathbf{r}} \hat{\theta}^{(a)} = \delta^{(a)}_B y^B , \quad (4.10)$$

$$\mathbf{i}_{\mathbf{r}} \omega^{(a)}_{(b)} = 0 , \quad (4.11)$$

where $\mathbf{i}_{\mathbf{r}}$ denotes the interior product [26] with respect to the vector field \mathbf{r} .⁸

The next step is to relate the second order Lie derivative of $\hat{\theta}^{(a)}$ and the curvature two-form. Using eqs. (A.58, 4.11) one writes,

$$\mathcal{L}_{\mathbf{r}}\hat{\theta}^{(a)} = \omega^{(a)}_{(b)}\mathbf{i}_{\mathbf{r}}\hat{\theta}^{(b)} + d\mathbf{i}_{\mathbf{r}}\hat{\theta}^{(a)} . \quad (4.14)$$

Then using the following result, which can be easily proved by using (4.10),

$$\mathcal{L}_{\mathbf{r}}d\mathbf{i}_{\mathbf{r}}\hat{\theta}^{(a)} = d\mathbf{i}_{\mathbf{r}}\hat{\theta}^{(a)} , \quad (4.15)$$

one arrives at,

$$\mathcal{L}_{\mathbf{r}}(\mathcal{L}_{\mathbf{r}} - 1)\hat{\theta}^{(a)} = \mathcal{L}_{\mathbf{r}}\omega^{(a)}_{(b)}\mathbf{i}_{\mathbf{r}}\hat{\theta}^{(b)} . \quad (4.16)$$

Finally, one calculates two sides of the above equation independently. The left hand side is directly calculated by noting,

$$\hat{\theta}^{(a)} = e^{(a)}_{\alpha}(x, y)dx^{\alpha} + e^{(a)}_A(x, y)dy^A , \quad (4.17)$$

and the right hand side can be calculated by using eqs.(A.59, 4.11). This leads to the following second order differential equation for the vielbein,

$$\mathbf{d}(\mathbf{d} + \epsilon^b)e^{(a)}_b(x, y) = \rho^{(a)}_{(c)}(x, y; y)e^{(c)}_b(x, y) , \quad (4.18)$$

where

$$\rho^{(a)}_{(b)}(x, y; \tilde{y}) \equiv r^{(a)}_{CD(b)}(x, y)\tilde{y}^C\tilde{y}^D . \quad (4.19)$$

For any function $f(x, y)$ we have defined,

$$\mathbf{d}f(x, y) = y^A\partial_A f(x, y) , \quad (4.20)$$

⁸To see how (4.10, 4.11) and (3.4) are equivalent one first shows that (3.4) implies,

$$\gamma^a_{BC}(x, y)y^By^C = 0 , \quad (4.12)$$

and *vice versa*. One then rewrites the above equations in terms of the vielbein and spin connection coefficients to derive (4.11) and,

$$\partial_A e^{(a)}_B(x, y)y^Ay^B = 0 , \quad (4.13)$$

which, in turn, is solved by (4.10).

and,

$$\epsilon^b = \begin{cases} 1 & \text{when } b = B , \\ -1 & \text{when } b = \beta . \end{cases} \quad (4.21)$$

Following [22], we Taylor expand both sides of (4.18) around $y = 0$ and equate the n -th order terms. This gives,

$$\begin{aligned} \mathbf{d}_0^n e^{(a)}{}_B(x, 0) &= \frac{n-1}{n+1} \mathbf{d}_0^{n-2} [\rho^{(a)}{}_{(c)}(x, 0; y) e^{(c)}{}_B(x, 0)] , \quad n \geq 1 , \\ \mathbf{d}_0^n e^{(a)}{}_\beta(x, 0) &= \mathbf{d}_0^{n-2} [\rho^{(a)}{}_{(c)}(x, 0; y) e^{(c)}{}_\beta(x, 0)] , \quad n \geq 2 , \end{aligned} \quad (4.22)$$

where we have introduced a new notation,

$$\mathbf{d}_0^n [f(x, 0; y) g(x, 0)] = y^{A_1} \cdots y^{A_n} \lim_{\tilde{y} \rightarrow 0} \frac{\partial}{\partial \tilde{y}^{A_1}} \cdots \frac{\partial}{\partial \tilde{y}^{A_n}} [f(x, \tilde{y}; y) g(x, \tilde{y})] . \quad (4.23)$$

The integral forms that solve the above equations are as follows,

$$\begin{aligned} e^{(a)}{}_B(x, y) &= \delta^{(a)}{}_B + \int_0^1 dt F_1(t) \rho^{(a)}{}_{(c)}(x, ty; y) e^{(c)}{}_B(x, ty) , \\ e^{(a)}{}_\beta(x, y) &= \bar{e}^{(a)}{}_\beta(x) + y^A \bar{e}^{(a)}{}_{\beta, A}(x) + \int_0^1 dt F_3(t) \rho^{(a)}{}_{(c)}(x, ty; y) e^{(c)}{}_\beta(x, ty) . \end{aligned} \quad (4.24)$$

The first term in the first equation and the first two terms in the second equation are “initial conditions”. The former is obtained by contracting the first equation with y^B , using (4.10) and then noticing that the integral term does not contribute under such a contraction because of the anti-symmetry property of the Riemann tensor. For the latter we have⁹,

$$\bar{e}^{(\alpha)}{}_\beta = E^{(\alpha)}{}_\beta(x) , \quad \bar{e}^{(A)}{}_\beta = 0 , \quad y^A \bar{e}^{(a)}{}_{\beta, A} = y^A \bar{\omega}_\beta^{(a)}{}_A , \quad (4.25)$$

where $E^{(\alpha)}{}_\beta$ is the vielbein of the induced metric $G_{\alpha\beta}$ on the submanifold. While the first equation is obvious, the second equation is required by consistency with (3.4). The third equation is obtained as follows. Using the following relation (*total covariant derivative*) ,

$$\partial_a e^{(b)}{}_c - \gamma_{ac}^d e^{(b)}{}_d + \omega_a^{(b)}{}_{(c)} e^{(e)}{}_c = 0 , \quad (4.26)$$

⁹In the rest of the discussion in this subsection, we will refrain from explicitly writing down the arguments. It is understood that a geometric quantity denoted by a lower case symbol without a bar has an argument (x, y) and the same with a bar has an argument (x) .

and (4.10) one gets on the submanifold,

$$\bar{\gamma}_{\beta A}^d \bar{e}^{(a)}{}_d = \bar{\gamma}_{A\beta}^d \bar{e}^{(a)}{}_d = \bar{\omega}_\beta^{(a)}{}_A , \quad (4.27)$$

Then using (4.26), (4.27) and (4.11) one arrives at the third equation in (4.25).

4.2 Closed form expressions for curvature expansion-coefficients

We write,

$$e^{(a)}{}_b(x, y) = \sum_{p=0}^{\infty} e_p^{(a)}{}_b(x, y) , \quad (4.28)$$

where e_p is the contribution at p -th order in curvature, such that,

$$\begin{aligned} e_0^{(a)}{}_B &= \delta^{(a)}{}_B , \\ e_0^{(a)}{}_\beta &= \begin{cases} E^{(\alpha)}{}_\beta + \bar{\omega}_\beta^{(\alpha)}{}_C y^C , & \text{for } a = \alpha , \\ \bar{\omega}_\beta^{(A)}{}_C y^C , & \text{for } a = A . \end{cases} \end{aligned} \quad (4.29)$$

Following the same techniques as in [22] we find,

$$\begin{aligned} e_p^{(a)}{}_B(x, y) &= \sum_{s_1, \dots, s_p} \mathcal{F}_1^{(p)}(s_1, s_2, \dots, s_p) [(y \cdot \nabla)^{s_1} \rho(x, 0; y) \cdots (y \cdot \nabla)^{s_p} \rho(x, 0; y)]^{(a)}{}_{(b)} e_0^{(b)}{}_B , \\ e_p^{(a)}{}_\beta(x, y) &= \sum_{s_1, \dots, s_p} \mathcal{F}_3^{(p)}(s_1, s_2, \dots, s_p) [(y \cdot \nabla)^{s_1} \rho(x, 0; y) \cdots (y \cdot \nabla)^{s_p} \rho(x, 0; y)]^{(a)}{}_{(b)} e_0^{(b)}{}_\beta , \end{aligned} \quad (4.30)$$

where,

$$\begin{aligned} \mathcal{F}_q^{(p)}(s_1, s_2, \dots, s_p) &= \frac{1}{s_1! \cdots s_p!} \int_0^1 dt_1 t_1^{s_1 + \dots + s_p + 2p - 2} F_q(t_1) \int_0^1 dt_2 t_2^{s_2 + \dots + s_p + 2p - 4} F_q(t_2) \\ &\quad \cdots \int_0^1 dt_p t_p^{s_p} F_q(t_p) , \quad q = 1, 3 , \end{aligned} \quad (4.31)$$

and $F_1(t)$ and $F_3(t)$ are defined in eqs.(3.7). Explicit calculations give the following results,

$$\begin{aligned} \mathcal{F}_1^{(p)}(s_1, s_2, \dots, s_p) &= \frac{C_1^{(p)}(s_1, s_2, \dots, s_p)}{(s_1 + s_2 + \dots + s_p + 2p + 1)!} , \\ \mathcal{F}_3^{(p)}(s_1, s_2, \dots, s_p) &= \frac{C_3^{(p)}(s_1, s_2, \dots, s_p)}{(s_1 + s_2 + \dots + s_p + 2p)!} , \end{aligned} \quad (4.32)$$

where,

$$\begin{aligned} C_1^{(p)}(s_1, s_2, \dots, s_p) &= C_{s_1}^{s_1+s_2+\dots+s_p+2p-1} C_{s_2}^{s_2+s_3+\dots+s_p+2p-3} \dots C_{s_p}^{s_p+1} , \\ C_3^{(p)}(s_1, s_2, \dots, s_p) &= C_{s_1}^{s_1+s_2+\dots+s_p+2p-2} C_{s_2}^{s_2+s_3+\dots+s_p+2p-4} \dots 1 , \end{aligned} \quad (4.33)$$

C_r^n being the binomial coefficients.

In order to present the results in convenient matrix forms we introduce $d \times d$ matrices \mathbb{E}_0^\parallel , \mathbb{E}_0^\perp and $R_{2+s}(x, y)$ such that their elements are given as follows,

$$[\mathbb{E}_0^\parallel]_{a\beta} = e_0^{(a)}{}_\beta , \quad [\mathbb{E}_0^\parallel]_{aB} = 0 , \quad (4.34)$$

$$[\mathbb{E}_0^\perp]_{a\beta} = 0 , \quad [\mathbb{E}_0^\perp]_{aB} = e_0^{(a)}{}_B , \quad (4.35)$$

$$[R_{2+s}(x, y)]_{ab} = (y \cdot \nabla)^s \rho^{(a)}{}_{(b)}(x, 0; y) . \quad (4.36)$$

Notice that R_{2+s} is linear in curvature, but $(s+2)$ -th order in y . Using these we further define the following two matrices:

$$\mathbb{E}_q(x, y) = \mathbb{I} + \sum_{p=1}^{\infty} \sum_{s_1, \dots, s_p \geq 0} \mathcal{F}_q^{(p)}(s_1, \dots, s_p) R_{2+s_1}(x, y) \dots R_{2+s_p}(x, y) , \quad q = 1, 3 , \quad (4.37)$$

where \mathbb{I} is the $d \times d$ identity matrix. The vielbein matrix is given by,

$$\begin{aligned} \mathbb{E}(x, y) &= \mathbb{E}^\parallel(x, y) + \mathbb{E}^\perp(x, y) , \\ \mathbb{E}^\parallel(x, y) &= \mathbb{E}_3(x, y) \mathbb{E}_0^\parallel , \quad \mathbb{E}^\perp(x, y) = \mathbb{E}_1(x, y) \mathbb{E}_0^\perp , \end{aligned} \quad (4.38)$$

such that the nonzero elements of $\mathbb{E}^\parallel(x, y)$ and $\mathbb{E}^\perp(x, y)$ are given by $e^{(a)}{}_\beta(x, y)$ and $e^{(a)}{}_B(x, y)$ respectively.

The metric is given by,

$$g_{ab}(x, y) = e^{(a')}{}_a(x, y) \eta_{(a'b')} e^{(b')}{}_b(x, y) . \quad (4.39)$$

In matrix form we write,

$$g(x, y) = \mathbb{E}^T(x, y) \eta \mathbb{E}(x, y) = g^\parallel + h + h^T + g^\perp , \quad (4.40)$$

where

$$\begin{aligned} g^\parallel &= (\mathbb{E}^\parallel)^T \eta \mathbb{E}^\parallel = (\mathbb{E}_0^\parallel)^T G^\parallel \mathbb{E}_0^\parallel , \\ h &= (\mathbb{E}^\parallel)^T \eta \mathbb{E}^\perp = (\mathbb{E}_0^\parallel)^T H \mathbb{E}_0^\perp , \end{aligned}$$

$$g^\perp = (\mathbb{E}^\perp)^T \eta \mathbb{E}^\perp = (\mathbb{E}_0^\perp)^T G^\perp \mathbb{E}_0^\perp , \quad (4.41)$$

are all $d \times d$ matrices whose non-zero elements are,

$$[g^\parallel]_{\alpha\beta} = g_{\alpha\beta}(x, y) , \quad [h]_{\alpha B} = g_{\alpha B}(x, y) , \quad [g^\perp]_{AB} = g_{AB}(x, y) . \quad (4.42)$$

Equations (4.38) and (4.41) imply,

$$G^\parallel = \bar{\mathbb{E}}_3^T \mathbb{E}_3 , \quad H = \bar{\mathbb{E}}_3^T \mathbb{E}_1 , \quad G^\perp = \bar{\mathbb{E}}_1^T \mathbb{E}_1 , \quad (4.43)$$

where,

$$\bar{\mathbb{E}}_q = \eta \mathbb{E}_q . \quad (4.44)$$

We will display explicit results for the matrices \mathbb{E}_1 , \mathbb{E}_3 , G^\parallel , H and G^\perp up to 10-th order in y in appendix B.

5 Alternative proof of Florides-Synge theorem

Here we prove the integral theorem in (3.5) starting from our result in (4.18). Analogous integral equations for vielbein, namely (4.24), were derived from a set of equations (4.22) describing all the necessary transverse derivatives of the vielbein evaluated on the sub-manifold. We will first derive the analogue of eqs.(4.22) for the metric (eqs.(5.53) below). The integral equations in (3.5) will then follow as solutions to eqs.(5.53).

Writing $g_{ab} = e_a \cdot e_b = \eta_{(cd)} e_a^{(c)} e_b^{(d)}$ and using (4.18) we get,

$$\mathbf{d}^2 g_{ab} + \overset{a}{\epsilon} \mathbf{d} e_a \cdot e_b + \overset{b}{\epsilon} e_a \cdot \mathbf{d} e_b = 2(r_{aCDb} y^C y^D + \mathbf{d} e_a \cdot \mathbf{d} e_b) . \quad (5.45)$$

Next we show that the right hand side receives contribution only from the linear part of the curvature tensor, i.e.,

$$r_{aCDb} y^C y^D + \mathbf{d} e_a \cdot \mathbf{d} e_b = l_{aCDb} y^C y^D , \quad (5.46)$$

where l_{abcd} is as defined in eq.(3.8). To establish the above equation we first write,

$$r_{abcd} = l_{abcd} + q_{abcd} , \quad (5.47)$$

where the quadratic part of the curvature tensor is given by,

$$q_{abcd} = g_{ef} (\gamma_{ad}^e \gamma_{bc}^f - \gamma_{ac}^e \gamma_{bd}^f) . \quad (5.48)$$

The condition in (4.12) implies that the first term does not contribute in the contraction $q_{aCDb}y^Cy^D$. The contribution from the second term can be calculated by using,

$$\partial_a g_{bC} y^C = (\eta_{bC} - g_{bC}) \delta_a^C, \quad (5.49)$$

which follows from eq.(3.4). The result is,

$$\begin{aligned} q_{aCDb} y^C y^D &= -\frac{1}{4} g^{cd} (\mathbf{d}g_{ca} + g_{aC} \delta_c^C - g_{cC} \delta_a^C) (\mathbf{d}g_{db} + g_{bD} \delta_d^D - g_{dD} \delta_b^D), \\ &= -\mathbf{d}e_a \cdot \mathbf{d}e_b, \end{aligned} \quad (5.50)$$

establishing (5.46). To arrive at the second line we used the following result,

$$\begin{aligned} \mathbf{d}e_a \cdot e_b - e_a \cdot \mathbf{d}e_b &= y^C (g_{bd} \gamma_{Ca}^d - g_{ad} \gamma_{Cb}^d), \\ &= g_{aC} \delta_b^C - g_{bC} \delta_a^C, \end{aligned} \quad (5.51)$$

where the first line is obtained by using an analogue of eq.(4.26) with indices properly placed and (4.11). The second line results from a direct calculation using the explicit form of the Christoffel symbols in terms of metric.

Using (5.46) and (5.51) in (5.45) we now rewrite the relevant second order differential equations for various metric components in the following way,

$$\begin{aligned} \mathbf{d}^2 g_{AB} + \mathbf{d}g_{AB} &= 2y^C y^D l_{ACDB}, \\ \mathbf{d}^2 g_{A\beta} - g_{A\beta} &= 2y^C y^D l_{ACD\beta}, \\ \mathbf{d}^2 g_{\alpha\beta} - \mathbf{d}g_{\alpha\beta} &= 2y^C y^D l_{\alpha CD\beta}. \end{aligned} \quad (5.52)$$

Following the method of §4.1, we Taylor expand each of the above equations around $y = 0$ to find the analogues of eqs.(4.22). The results are as follows,

$$\begin{aligned} \mathbf{d}_0^n g_{AB}(x, 0) &= 2 \frac{n-1}{n+1} y^C y^D \mathbf{d}_0^{n-2} l_{ACDB}(x, 0), \quad n \geq 1, \\ \mathbf{d}_0^n g_{A\beta}(x, 0) &= \frac{2n}{n+1} y^C y^D \mathbf{d}_0^{n-2} l_{ACD\beta}(x, 0), \quad n \geq 2, \\ \mathbf{d}_0^n g_{\alpha\beta}(x, 0) &= 2y^C y^D \mathbf{d}_0^{n-2} l_{\alpha CD\beta}(x, 0), \quad n \geq 2. \end{aligned} \quad (5.53)$$

The integral equations that solve the above equations are precisely the ones in eqs.(3.5).

A Notations and conventions

We denote the coordinate and non-coordinate bases of the tangent space by \mathcal{E}_a and $\hat{\mathcal{E}}_{(a)}$ respectively,

$$\mathcal{E}_b = e^{(a)}{}_b \hat{\mathcal{E}}_{(a)} , \quad (\text{A.54})$$

$e^{(a)}{}_b$ being the vielbein-components. We denote by η the diagonal matrix with indicators of the non-coordinate basis as diagonal elements,

$$\eta_{ab} = \langle \hat{\mathcal{E}}_{(a)}, \hat{\mathcal{E}}_{(b)} \rangle . \quad (\text{A.55})$$

The bases dual to \mathcal{E}_a and $\hat{\mathcal{E}}_{(a)}$ are denoted by dz^a and $\hat{\theta}^{(a)}$ respectively. We consider the torsion-less situation and denote the Christoffel symbols by γ^a_{bc} ,

$$\nabla_b \mathcal{E}_c \equiv \nabla_{\mathcal{E}_b} \mathcal{E}_c = \gamma^a_{bc} \mathcal{E}_a , \quad (\text{A.56})$$

where ∇ is the Levi-Civita connection. The connection one-form $\omega^{(b)}_{(c)} = \omega_{(a)}^{(b)}{}_{(c)} \hat{\theta}^{(a)}$ is given by,

$$\nabla_{(a)} \mathcal{E}_{(c)} \equiv \nabla_{\hat{\mathcal{E}}_{(a)}} \mathcal{E}_{(c)} = \omega_{(a)}^{(b)}{}_{(c)} \mathcal{E}_{(b)} . \quad (\text{A.57})$$

Cartan's structure equations in our case are given by,

$$d\hat{\theta}^{(a)} + \omega^{(a)}{}_{(b)} \wedge \hat{\theta}^{(b)} = 0 , \quad (\text{A.58})$$

$$d\omega^{(a)}{}_{(b)} + \omega^{(a)}{}_{(c)} \wedge \omega^{(c)}{}_{(b)} = r^{(a)}{}_{(b)} , \quad (\text{A.59})$$

where $r^{(a)}{}_{(b)} = \frac{1}{2} r^{(a)}{}_{(b)(c)(d)} \hat{\theta}^{(c)} \wedge \hat{\theta}^{(d)}$ is the curvature two-form,

$$r^{(a)}{}_{(b)(c)(d)} = \langle \hat{\theta}^{(a)}, (\nabla_{(c)} \nabla_{(d)} - \nabla_{(d)} \nabla_{(c)} - \nabla_{[\hat{\mathcal{E}}_{(c)}, \hat{\mathcal{E}}_{(d)}]}) \hat{\mathcal{E}}_{(b)} \rangle , \quad (\text{A.60})$$

$[\cdot, \cdot]$ being the Lie bracket [26].

B Coefficients up to tenth order

Here we will display the expansion coefficients up to 10-th order in y for the matrices \mathbb{E}_1 , \mathbb{E}_3 , G^\parallel , H and G^\perp . We first expand these matrices in the following way,

$$\mathbb{E}_q = \mathbb{I} + \sum_{p=2}^{\infty} \mathbb{E}_q^{(p)}(x, y) , \quad q = 1, 3 ,$$

$$\begin{aligned}
G^{\parallel} &= \eta + \sum_{p=2}^{\infty} G^{\parallel(p)} = \eta + \sum_{p=2}^{\infty} (\bar{\mathbb{E}}_3^{(p)} + \bar{\mathbb{E}}_3^{(p)T}) + \sum_{p,q=2}^{\infty} \bar{\mathbb{E}}_3^{(p)T} \mathbb{E}_3^{(q)} , \\
H &= \eta + \sum_{p=2}^{\infty} H^{(p)} = \eta + \sum_{p=2}^{\infty} (\bar{\mathbb{E}}_1^{(p)} + \bar{\mathbb{E}}_3^{(p)T}) + \sum_{p,q=2}^{\infty} \bar{\mathbb{E}}_3^{(p)T} \mathbb{E}_1^{(q)} , \\
G^{\perp} &= \eta + \sum_{p=2}^{\infty} G^{\perp(p)} = \eta + \sum_{p=2}^{\infty} (\bar{\mathbb{E}}_1^{(p)} + \bar{\mathbb{E}}_1^{(p)T}) + \sum_{p,q=2}^{\infty} \bar{\mathbb{E}}_1^{(p)T} \mathbb{E}_1^{(q)} , \tag{B.61}
\end{aligned}$$

where $\mathbb{E}_q^{(p)}(x, y)$, $G^{\parallel(p)}(x, y)$, $H^{(p)}(x, y)$ and $G^{\perp(p)}(x, y)$ are contributions to the relevant matrices at the p -th order in y . The results up to $p = 10$ are given below. The following notation will be used in writing down results for G^{\parallel} , H and G^{\perp} . Given a matrix of the form: $\bar{R}_m R_n \cdots = \eta R_m R_n \cdots$, we define,

$$\{\bar{R}_m R_n \cdots\} = \eta R_m R_n \cdots + \eta \cdots R_n R_m . \tag{B.62}$$

Results for \mathbb{E}_1 :

$$\begin{aligned}
\mathbb{E}_1^{(2)} &= \frac{1}{6} R_2 , \\
\mathbb{E}_1^{(3)} &= \frac{1}{12} R_3 , \\
\mathbb{E}_1^{(4)} &= \frac{1}{40} R_4 + \frac{1}{120} R_2^2 , \\
\mathbb{E}_1^{(5)} &= \frac{1}{180} R_5 + \frac{1}{180} R_3 R_2 + \frac{1}{360} R_2 R_3 , \\
\mathbb{E}_1^{(6)} &= \frac{1}{1008} R_6 + \frac{1}{504} R_4 R_2 + \frac{1}{1680} R_2 R_4 + \frac{1}{504} R_3^2 + \frac{1}{5040} R_2^3 , \\
\mathbb{E}_1^{(7)} &= \frac{1}{6720} R_7 + \frac{1}{2016} R_5 R_2 + \frac{1}{1344} R_4 R_3 + \frac{1}{2240} R_3 R_4 + \frac{1}{6720} R_3 R_2^2 + \frac{1}{10080} R_2 R_5 \\
&\quad + \frac{1}{10080} R_2 R_3 R_2 + \frac{1}{20160} R_2^2 R_3 , \\
\mathbb{E}_1^{(8)} &= \frac{1}{51840} R_8 + \frac{1}{10368} R_6 R_2 + \frac{1}{5184} R_5 R_3 + \frac{1}{5760} R_4^2 + \frac{1}{17280} R_4 R_2^2 \\
&\quad + \frac{1}{12960} R_3 R_5 + \frac{1}{12960} R_3^2 R_2 + \frac{1}{25920} R_3 R_2 R_3 + \frac{1}{72576} R_2 R_6 \\
&\quad + \frac{1}{36288} R_2 R_4 R_2 + \frac{1}{36288} R_2 R_3^2 + \frac{1}{120960} R_2^2 R_4 + \frac{1}{362880} R_2^4 , \\
\mathbb{E}_1^{(9)} &= \frac{1}{453600} R_9 + \frac{1}{64800} R_7 R_2 + \frac{1}{25920} R_6 R_3 + \frac{1}{21600} R_5 R_4 + \frac{1}{64800} R_5 R_2^2 + \frac{1}{32400} R_4 R_5 \\
&\quad + \frac{1}{32400} R_4 R_3 R_2 + \frac{1}{64800} R_4 R_2 R_3 + \frac{1}{90720} R_3 R_6 + \frac{1}{45360} R_3 R_4 R_2 + \frac{1}{45360} R_3^3 \\
&\quad + \frac{1}{151200} R_3 R_2 R_4 + \frac{1}{453600} R_3 R_2^3 + \frac{1}{604800} R_2 R_7 + \frac{1}{181440} R_2 R_5 R_2 + \frac{1}{120960} R_2 R_4 R_3
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{201600} R_2 R_3 R_4 + \frac{1}{604800} R_2 R_3 R_2^2 + \frac{1}{907200} R_2^2 R_5 + \frac{1}{907200} R_2^2 R_3 R_2 \\
& + \frac{1}{1814400} R_2^3 R_3 , \\
\mathbb{E}_1^{(10)} = & \frac{1}{4435200} R_{10} + \frac{1}{475200} R_8 R_2 + \frac{1}{158400} R_7 R_3 + \frac{1}{105600} R_6 R_4 + \frac{1}{316800} R_6 R_2^2 \\
& + \frac{1}{118800} R_5^2 + \frac{1}{118800} R_5 R_3 R_2 + \frac{1}{237600} R_5 R_2 R_3 + \frac{1}{221760} R_4 R_6 \\
& + \frac{1}{110880} R_4^2 R_2 + \frac{1}{110880} R_4 R_3^2 + \frac{1}{369600} R_4 R_2 R_4 + \frac{1}{1108800} R_4 R_2^3 + \frac{1}{739200} R_3 R_7 \\
& + \frac{1}{221760} R_3 R_5 R_2 + \frac{1}{147840} R_3 R_4 R_3 + \frac{1}{246400} R_3^2 R_4 + \frac{1}{739200} R_3^2 R_2^2 + \frac{1}{1108800} R_3 R_2 R_5 \\
& + \frac{1}{1108800} R_3 R_2 R_3 R_2 + \frac{1}{2217600} R_3 R_2^2 R_3 + \frac{1}{5702400} R_2 R_8 \\
& + \frac{1}{1140480} R_2 R_6 R_2 + \frac{1}{570240} R_2 R_5 R_3 + \frac{1}{633600} R_2 R_4^2 \\
& + \frac{1}{1900800} R_2 R_4 R_2^2 + \frac{1}{1425600} R_2 R_3 R_5 + \frac{1}{1425600} R_2 R_3^2 R_2 \\
& + \frac{1}{2851200} R_2 R_3 R_2 R_3 + \frac{1}{7983360} R_2^2 R_6 + \frac{1}{3991680} R_2^2 R_4 R_2 \\
& + \frac{1}{3991680} R_2^2 R_3^2 + \frac{1}{13305600} R_2^3 R_4 + \frac{1}{39916800} R_2^5 .
\end{aligned}$$

Results for \mathbb{E}_3 :

$$\begin{aligned}
\mathbb{E}_3^{(2)} &= \frac{1}{2} R_2 , \\
\mathbb{E}_3^{(3)} &= \frac{1}{6} R_3 , \\
\mathbb{E}_3^{(4)} &= \frac{1}{24} R_4 + \frac{1}{24} R_2^2 , \\
\mathbb{E}_3^{(5)} &= \frac{1}{120} R_5 + \frac{1}{40} R_3 R_2 + \frac{1}{120} R_2 R_3 , \\
\mathbb{E}_3^{(6)} &= \frac{1}{720} R_6 + \frac{1}{120} R_4 R_2 + \frac{1}{720} R_2 R_4 + \frac{1}{180} R_3^2 + \frac{1}{720} R_2^3 , \\
\mathbb{E}_3^{(7)} &= \frac{1}{5040} R_7 + \frac{1}{504} R_5 R_2 + \frac{1}{504} R_4 R_3 + \frac{1}{1008} R_3 R_4 + \frac{1}{1008} R_3 R_2^2 \\
& + \frac{1}{5040} R_2 R_5 + \frac{1}{1680} R_2 R_3 R_2 + \frac{1}{5040} R_2^2 R_3 , \\
\mathbb{E}_3^{(8)} &= \frac{1}{40320} R_8 + \frac{1}{2688} R_6 R_2 + \frac{1}{2016} R_5 R_3 + \frac{1}{2688} R_4^2 + \frac{1}{2688} R_4 R_2^2 \\
& + \frac{1}{6720} R_3 R_5 + \frac{1}{2240} R_3^2 R_2 + \frac{1}{6720} R_3 R_2 R_3 + \frac{1}{40320} R_2 R_6 \\
& + \frac{1}{6720} R_2 R_4 R_2 + \frac{1}{10080} R_2 R_3^2 + \frac{1}{40320} R_2^2 R_4 + \frac{1}{40320} R_2^4 , \\
\mathbb{E}_3^{(9)} &= \frac{1}{362880} R_9 + \frac{1}{17280} R_7 R_2 + \frac{1}{10368} R_6 R_3 + \frac{1}{10368} R_5 R_4 + \frac{1}{10368} R_5 R_2^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{17280} R_4 R_5 + \frac{1}{5760} R_4 R_3 R_2 + \frac{1}{17280} R_4 R_2 R_3 + \frac{1}{51840} R_3 R_6 \\
& + \frac{1}{8640} R_3 R_4 R_2 + \frac{1}{12960} R_3^3 + \frac{1}{51840} R_3 R_2 R_4 + \frac{1}{51840} R_3 R_2^3 \\
& + \frac{1}{362880} R_2 R_7 + \frac{1}{36288} R_2 R_5 R_2 + \frac{1}{36288} R_2 R_4 R_3 + \frac{1}{72576} R_2 R_3 R_4 \\
& + \frac{1}{72576} R_2 R_3 R_2^2 + \frac{1}{362880} R_2^2 R_5 + \frac{1}{120960} R_2^2 R_3 R_2 \\
& + \frac{1}{362880} R_2^3 R_3 , \\
\mathbb{E}_3^{(10)} = & \frac{1}{3628800} R_{10} + \frac{1}{129600} R_8 R_2 + \frac{1}{64800} R_7 R_3 + \frac{1}{51840} R_6 R_4 + \frac{1}{51840} R_6 R_2^2 \\
& + \frac{1}{64800} R_5^2 + \frac{1}{21600} R_5 R_3 R_2 + \frac{1}{64800} R_5 R_2 R_3 + \frac{1}{129600} R_4 R_6 \\
& + \frac{1}{21600} R_4^2 R_2 + \frac{1}{32400} R_4 R_3^2 + \frac{1}{129600} R_4 R_2 R_4 \\
& + \frac{1}{129600} R_4 R_2^3 + \frac{1}{453600} R_3 R_7 + \frac{1}{45360} R_3 R_5 R_2 \\
& + \frac{1}{45360} R_3 R_4 R_3 + \frac{1}{90720} R_3^2 R_4 + \frac{1}{90720} R_3^2 R_2^2 \\
& + \frac{1}{453600} R_3 R_2 R_5 + \frac{1}{151200} R_3 R_2 R_3 R_2 + \frac{1}{453600} R_3 R_2^2 R_3 \\
& + \frac{1}{3628800} R_2 R_8 + \frac{1}{241920} R_2 R_6 R_2 + \frac{1}{181440} R_2 R_5 R_3 \\
& + \frac{1}{241920} R_2 R_4^2 + \frac{1}{241920} R_2 R_4 R_2^2 + \frac{1}{604800} R_2 R_3 R_5 \\
& + \frac{1}{201600} R_2 R_3^2 R_2 + \frac{1}{604800} R_2 R_3 R_2 R_3 + \frac{1}{3628800} R_2^2 R_6 \\
& + \frac{1}{604800} R_2^2 R_4 R_2 + \frac{1}{907200} R_2^2 R_3^2 + \frac{1}{3628800} R_2^3 R_4 \\
& + \frac{1}{3628800} R_2^5 .
\end{aligned}$$

Results for G^{\parallel} :

$$\begin{aligned}
G^{\parallel(2)} &= \bar{R}_2 , \\
G^{\parallel(3)} &= \frac{1}{3} \bar{R}_3 , \\
G^{\parallel(4)} &= \frac{1}{12} \bar{R}_4 + \frac{1}{3} \bar{R}_2 R_2 , \\
G^{\parallel(5)} &= \frac{1}{60} \bar{R}_5 + \frac{1}{60} \{ \bar{R}_3 R_2 \} , \\
G^{\parallel(6)} &= \frac{1}{360} \bar{R}_6 + \frac{11}{360} \{ \bar{R}_4 R_2 \} + \frac{7}{180} \bar{R}_3 R_3 + \frac{2}{45} \bar{R}_2 R_2^2 , \\
G^{\parallel(7)} &= \frac{1}{2520} \bar{R}_7 + \frac{2}{315} \{ \bar{R}_5 R_2 \} + \frac{5}{504} \{ \bar{R}_4 R_3 \} + \frac{31}{2520} \{ \bar{R}_3 R_2^2 \} + \frac{11}{420} \bar{R}_2 R_3 R_2 ,
\end{aligned}$$

$$\begin{aligned}
G^{\parallel(8)} &= \frac{1}{20160} \bar{R}_8 + \frac{11}{10080} \{\bar{R}_6 R_2\} + \frac{41}{20160} \{\bar{R}_5 R_3\} + \frac{5}{2016} \bar{R}_4 R_4 + \frac{19}{6720} \{\bar{R}_4 R_2^2\} \\
&\quad + \frac{41}{20160} \bar{R}_3 R_5 + \frac{151}{20160} \{\bar{R}_3 R_3 R_2\} + \frac{31}{10080} \bar{R}_3 R_2 R_3 + \frac{29}{3360} \bar{R}_2 R_4 R_2 + \frac{1}{315} \bar{R}_2 R_2^3, \\
G^{\parallel(9)} &= \frac{1}{181440} \bar{R}_9 + \frac{29}{181440} \{\bar{R}_7 R_2\} + \frac{1}{2880} \{\bar{R}_6 R_3\} + \frac{13}{25920} \{\bar{R}_5 R_4\} + \frac{11}{20160} \{\bar{R}_5 R_2^2\} \\
&\quad + \frac{1}{181440} \{\bar{R}_4 R_3 R_2\} + \frac{17}{25920} \{\bar{R}_4 R_2 R_3\} + \frac{229}{90720} \{\bar{R}_3 R_4 R_2\} + \frac{13}{6480} \bar{R}_3 R_3^2 \\
&\quad + \frac{127}{181440} \{\bar{R}_3 R_2^3\} + \frac{37}{18144} \bar{R}_2 R_5 R_2 + \frac{337}{181440} \{\bar{R}_2 R_3 R_2^2\} \\
G^{\parallel(10)} &= \frac{1}{1814400} \bar{R}_{10} + \frac{37}{1814400} \{\bar{R}_8 R_2\} + \frac{23}{453600} \{\bar{R}_7 R_3\} + \frac{11}{129600} \{\bar{R}_6 R_4\} \\
&\quad + \frac{163}{1814400} \{\bar{R}_6 R_2^2\} + \frac{13}{129600} \bar{R}_5 R_5 + \frac{1}{3024} \{\bar{R}_5 R_3 R_2\} + \frac{109}{907200} \{\bar{R}_5 R_2 R_3\} \\
&\quad + \frac{353}{604800} \{\bar{R}_4 R_4 R_2\} + \frac{199}{453600} \{\bar{R}_4 R_3^2\} + \frac{17}{129600} \bar{R}_4 R_2 R_4 + \frac{247}{1814400} \{\bar{R}_4 R_2^3\} \\
&\quad + \frac{11}{18144} \{\bar{R}_3 R_5 R_2\} + \frac{2}{2835} \bar{R}_3 R_4 R_3 + \frac{13}{28350} \{\bar{R}_3 R_3 R_2^2\} + \frac{59}{151200} \bar{R}_3 R_2 R_3 R_2 \\
&\quad + \frac{127}{907200} \bar{R}_3 R_2^2 R_3 + \frac{23}{60480} \bar{R}_2 R_6 R_2 + \frac{53}{86400} \{\bar{R}_2 R_4 R_2^2\} + \frac{109}{100800} \bar{R}_2 R_3^2 R_2 \\
&\quad + \frac{59}{151200} \bar{R}_2 R_3 R_2 R_3 + \frac{2}{14175} \bar{R}_2 R_2^4.
\end{aligned}$$

Results for H :

$$\begin{aligned}
H^{(2)} &= \frac{2}{3} \bar{R}_2, \\
H^{(3)} &= \frac{1}{4} \bar{R}_3, \\
H^{(4)} &= \frac{1}{15} \bar{R}_4 + \frac{2}{15} \bar{R}_2 R_2, \\
H^{(5)} &= \frac{1}{72} \bar{R}_5 + \frac{1}{24} \bar{R}_3 R_2 + \frac{5}{72} \bar{R}_2 R_3, \\
H^{(6)} &= \frac{1}{420} \bar{R}_6 + \frac{1}{63} \bar{R}_4 R_2 + \frac{3}{140} \bar{R}_3 R_3 + \frac{3}{140} \bar{R}_2 R_4 + \frac{5}{504} \bar{R}_2 R_2^2, \\
H^{(7)} &= \frac{1}{2880} \bar{R}_7 + \frac{1}{480} \bar{R}_5 R_2 + \frac{1}{192} \bar{R}_4 R_3 + \frac{19}{2880} \bar{R}_3 R_4 + \frac{1}{320} \bar{R}_3 R_2^2 + \frac{7}{1440} \bar{R}_2 R_5 \\
&\quad + \frac{11}{1440} \bar{R}_2 R_3 R_2 + \frac{17}{2880} \bar{R}_2 R_2 R_3, \\
H^{(8)} &= \frac{1}{22680} \bar{R}_8 + \frac{1}{2835} \bar{R}_6 R_2 + \frac{47}{45360} \bar{R}_5 R_3 + \frac{1}{630} \bar{R}_4 R_4 + \frac{1}{1512} \bar{R}_4 R_2^2 \\
&\quad + \frac{17}{11340} \bar{R}_3 R_5 + \frac{23}{11340} \bar{R}_3 R_3 R_2 + \frac{61}{45360} \bar{R}_3 R_2 R_3 + \frac{1}{1134} \bar{R}_2 R_6 + \frac{29}{11340} \bar{R}_2 R_4 R_2 \\
&\quad + \frac{23}{6480} \bar{R}_2 R_3^2 + \frac{13}{7560} \bar{R}_2 R_2 R_4 + \frac{2}{2835} \bar{R}_2 R_2^3, \\
H^{(9)} &= \frac{1}{201600} \bar{R}_9 + \frac{31}{604800} \bar{R}_7 R_2 + \frac{1}{5760} \bar{R}_6 R_3 + \frac{1}{3200} \bar{R}_5 R_4 + \frac{73}{604800} \bar{R}_5 R_2^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{31}{86400} \bar{R}_4 R_5 + \frac{89}{201600} \bar{R}_4 R_3 R_2 + \frac{23}{86400} \bar{R}_4 R_2 R_3 + \frac{11}{40320} \bar{R}_3 R_6 + \frac{43}{60480} \bar{R}_3 R_4 R_2 \\
& + \frac{1}{1120} \bar{R}_3 R_3^2 + \frac{1}{2688} \bar{R}_3 R_2 R_4 + \frac{17}{120960} \bar{R}_3 R_2^3 + \frac{3}{22400} \bar{R}_2 R_7 + \frac{37}{60480} \bar{R}_2 R_5 R_2 \\
& + \frac{1}{840} \bar{R}_2 R_4 R_3 + \frac{23}{22400} \bar{R}_2 R_3 R_4 + \frac{79}{201600} \bar{R}_2 R_3 R_2^2 + \frac{229}{604800} \bar{R}_2 R_2 R_5 \\
& + \frac{31}{67200} \bar{R}_2 R_2 R_3 R_2 + \frac{167}{604800} \bar{R}_2 R_2^2 R_3 , \\
H^{(10)} = & \frac{1}{1995840} \bar{R}_{10} + \frac{13}{1995840} \bar{R}_8 R_2 + \frac{5}{199584} \bar{R}_7 R_3 + \frac{37}{712800} \bar{R}_6 R_4 + \frac{191}{9979200} \bar{R}_6 R_2^2 \\
& + \frac{1}{14256} \bar{R}_5 R_5 + \frac{1}{12320} \bar{R}_5 R_3 R_2 + \frac{23}{498960} \bar{R}_5 R_2 R_3 + \frac{13}{199584} \bar{R}_4 R_6 + \frac{1}{6336} \bar{R}_4 R_4 R_2 \\
& + \frac{37}{199584} \bar{R}_4 R_3^2 + \frac{349}{4989600} \bar{R}_4 R_2 R_4 + \frac{251}{9979200} \bar{R}_4 R_2^3 + \frac{83}{1995840} \bar{R}_3 R_7 \\
& + \frac{5}{28512} \bar{R}_3 R_5 R_2 + \frac{127}{399168} \bar{R}_3 R_4 R_3 + \frac{2477}{9979200} \bar{R}_3 R_3 R_4 + \frac{899}{9979200} \bar{R}_3 R_3 R_2^2 \\
& + \frac{79}{997920} \bar{R}_3 R_2 R_5 + \frac{1}{11088} \bar{R}_3 R_2 R_3 R_2 + \frac{101}{1995840} \bar{R}_3 R_2^2 R_3 + \frac{1}{57024} \bar{R}_2 R_8 \\
& + \frac{23}{199584} \bar{R}_2 R_6 R_2 + \frac{19}{66528} \bar{R}_2 R_5 R_3 + \frac{163}{475200} \bar{R}_2 R_4^2 + \frac{139}{1108800} \bar{R}_2 R_4 R_2^2 \\
& + \frac{1}{4455} \bar{R}_2 R_3 R_5 + \frac{257}{997920} \bar{R}_2 R_3^2 R_2 + \frac{29}{199584} \bar{R}_2 R_3 R_2 R_3 + \frac{1}{14784} \bar{R}_2 R_2 R_6 \\
& + \frac{65}{399168} \bar{R}_2 R_2 R_4 R_2 + \frac{19}{99792} \bar{R}_2 R_2 R_3^2 + \frac{713}{9979200} \bar{R}_2 R_2^2 R_4 + \frac{4}{155925} \bar{R}_2 R_2^4 ,
\end{aligned}$$

Results for G^\perp :

$$\begin{aligned}
G^{\perp(2)} &= \frac{1}{3} \bar{R}_2 , \\
G^{\perp(3)} &= \frac{1}{6} \bar{R}_3 , \\
G^{\perp(4)} &= \frac{1}{20} \bar{R}_4 + \frac{2}{45} \bar{R}_2 R_2 , \\
G^{\perp(5)} &= \frac{1}{90} \bar{R}_5 + \frac{1}{45} \{ \bar{R}_3 R_2 \} , \\
G^{\perp(6)} &= \frac{1}{504} \bar{R}_6 + \frac{17}{2520} \{ \bar{R}_4 R_2 \} + \frac{11}{1008} \bar{R}_3 R_3 + \frac{1}{315} \bar{R}_2^3 , \\
G^{\perp(7)} &= \frac{1}{3360} \bar{R}_7 + \frac{23}{15120} \{ \bar{R}_5 R_2 \} + \frac{11}{3360} \{ \bar{R}_4 R_3 \} + \frac{41}{30240} \{ \bar{R}_3 R_2^2 \} + \frac{31}{15120} \bar{R}_2 R_3 R_2 \\
G^{\perp(8)} &= \frac{1}{25920} \bar{R}_8 + \frac{5}{18144} \{ \bar{R}_6 R_2 \} + \frac{19}{25920} \{ \bar{R}_5 R_3 \} + \frac{7}{7200} \bar{R}_4 R_4 + \frac{113}{302400} \{ \bar{R}_4 R_2^2 \} \\
& + \frac{163}{181440} \{ \bar{R}_3 R_3 R_2 \} + \frac{7}{12960} \bar{R}_3 R_2 R_3 + \frac{13}{18144} \bar{R}_2 R_4 R_2 + \frac{2}{14175} \bar{R}_2 R_2^3 , \\
G^{\perp(9)} &= \frac{1}{226800} \bar{R}_9 + \frac{19}{453600} \{ \bar{R}_7 R_2 \} + \frac{1}{7560} \{ \bar{R}_6 R_3 \} + \frac{7}{32400} \{ \bar{R}_5 R_4 \} \\
& + \frac{1}{12600} \{ \bar{R}_5 R_2^2 \} + \frac{113}{453600} \bar{R}_4 R_3 R_2 + \frac{2}{14175} \bar{R}_4 R_2 R_3 + \frac{29}{90720} \bar{R}_3 R_4 R_2
\end{aligned}$$

$$\begin{aligned}
& + \frac{17}{45360} \bar{R}_3 R_3^2 + \frac{2}{14175} \bar{R}_3 R_2 R_4 + \frac{23}{453600} \{\bar{R}_3 R_2^3\} + \frac{1}{5670} \bar{R}_2 R_5 R_2 \\
& + \frac{29}{90720} \bar{R}_2 R_4 R_3 + \frac{113}{453600} \bar{R}_2 R_3 R_4 + \frac{41}{453600} \{\bar{R}_2 R_3 R_2^2\} , \\
G^{\perp(10)} = & \frac{1}{2217600} \bar{R}_{10} + \frac{47}{8553600} \{\bar{R}_8 R_2\} + \frac{89}{4435200} \{\bar{R}_7 R_3\} + \frac{43}{1108800} \{\bar{R}_6 R_4\} + \frac{829}{59875200} \{\bar{R}_6 R_2^2\} \\
& + \frac{17}{356400} \bar{R}_5 R_5 + \frac{113}{2138400} \bar{R}_5 R_3 R_2 + \frac{23}{798336} \bar{R}_5 R_2 R_3 + \frac{593}{6652800} \{\bar{R}_4 R_4 R_2\} \\
& + \frac{443}{4435200} \{\bar{R}_4 R_3^2\} + \frac{13}{369600} \bar{R}_4 R_2 R_4 + \frac{7}{570240} \{\bar{R}_4 R_2^3\} \\
& + \frac{191}{2395008} \bar{R}_3 R_5 R_2 + \frac{61}{443520} \bar{R}_3 R_4 R_3 + \frac{601}{17107200} \{\bar{R}_3 R_3 R_2^2\} + \frac{23}{798336} \bar{R}_3 R_2 R_5 \\
& + \frac{1879}{59875200} \bar{R}_3 R_2 R_3 R_2 + \frac{337}{19958400} \bar{R}_3 R_2^2 R_3 + \frac{29}{855360} \bar{R}_2 R_6 R_2 + \frac{191}{2395008} \bar{R}_2 R_5 R_3 \\
& + \frac{1889}{59875200} \bar{R}_2 R_4 R_2^2 + \frac{113}{2138400} \bar{R}_2 R_3 R_5 + \frac{31}{534600} \bar{R}_2 R_3^2 R_2 + \frac{1879}{59875200} \bar{R}_2 R_3 R_2 R_3 \\
& + \frac{1889}{59875200} \bar{R}_2 R_2 R_4 R_2 + \frac{2}{467775} \bar{R}_2 R_2^4 .
\end{aligned}$$

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